Automatic Control

“If you have a smart project, you can say "I'm an engineer"”

Lecture 6

Staff boarder

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Automatic Control
MPE 424

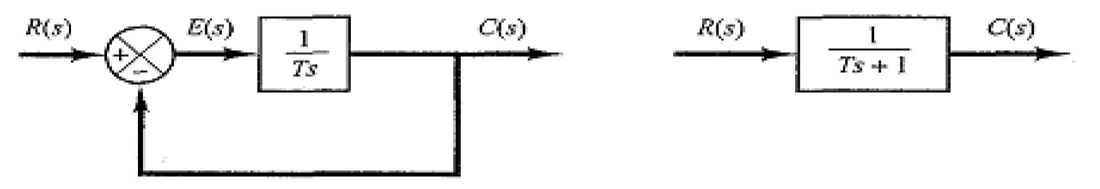
- **Lecture aims:**
  - Be familiar with the design formulas that relate the second-order pole locations to percent overshoot, settling time, rise time, and time to peak
  - Understand the concept of stability of dynamic systems

Time Response

Time response

- Consider the first-order system. Physically, this system may represent an *RC* circuit, thermal system, or the like. A simplified block diagram is shown in Figure. The input-output relationship is given by

$$\frac{C(s)}{R(s)} = \frac{1}{Ts + 1}$$



- In the following, we shall analyze the system responses to such inputs as the unit-step, unit-ramp, and unit-impulse functions. The initial conditions are assumed to be zero.

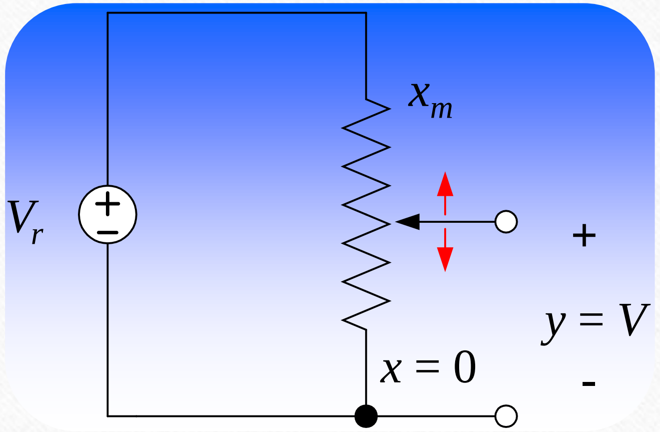
Zero-order Systems

The behavior is characterized by its static sensitivity, *K* and remains constant regardless of input frequency (ideal dynamic characteristic).

$$\boxed{a_0 y(t) = b_0 x(t)} \longrightarrow \boxed{y(t) = Kx(t)}$$

where K = static sensitivity = b_0/a_0

All the a's and b's other than a_0 and b_0 are zero.



$$V = V_r \cdot \frac{x}{x_m} \quad \text{here, } K = V_r / x_m$$

Where $0 \le x \le x_m$ and V_r is a reference voltage

A linear potentiometer used as position sensor is a zero-order sensor.

First-Order Systems

All the a's and b's other than a_1, a_0 and b_0 are zero.

$$a_1 \frac{dy(t)}{dt} + a_0 = b_0 x(t)$$

$$\boxed{\tau \frac{dy(t)}{dt} + y(t) = Kx(t)} \quad \Longleftrightarrow \quad \boxed{\frac{y}{x}(S) = \frac{K}{\tau S + 1}}$$

Where $K = b_0/a_0$ is the static sensitivity

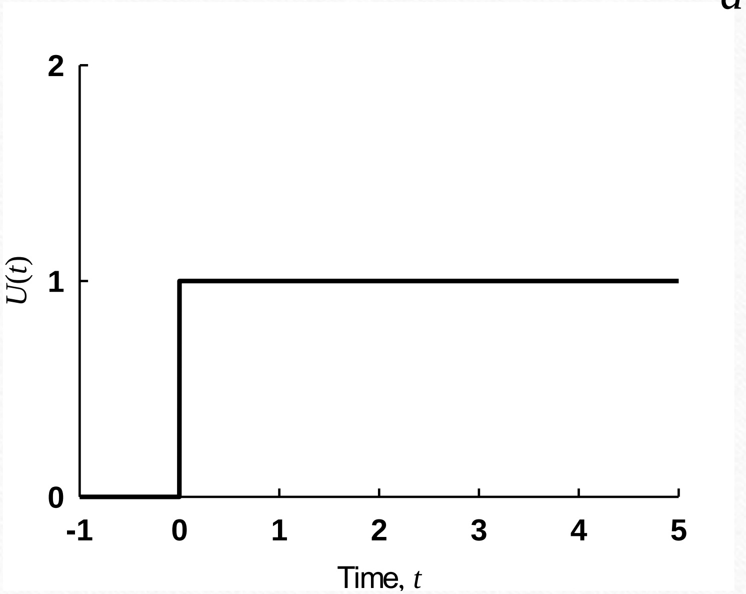
$\tau = a_1/a_0$ is the system's time constant (dimension of time)

First-Order Systems: Step Response

Assume for $t < 0$, $y = y_0$, at time = 0 the input quantity, x increases instantly by an amount A. Therefore $t > 0$

$$\tau \frac{dy(t)}{dt} + y(t) = KAU(t)$$

$$x(t) = AU(t) = \begin{cases} 0 & t \leq 0 \\ A & t > 0 \end{cases}$$



The complete solution:

$$\boxed{y(t) = Ce^{-t/\tau} + KA}$$

$$\underbrace{\phantom{Ce^{-t/\tau}}}_{y_{ocf}} \quad \underbrace{}_{y_{opi}}$$

Transient response **Steady state response**

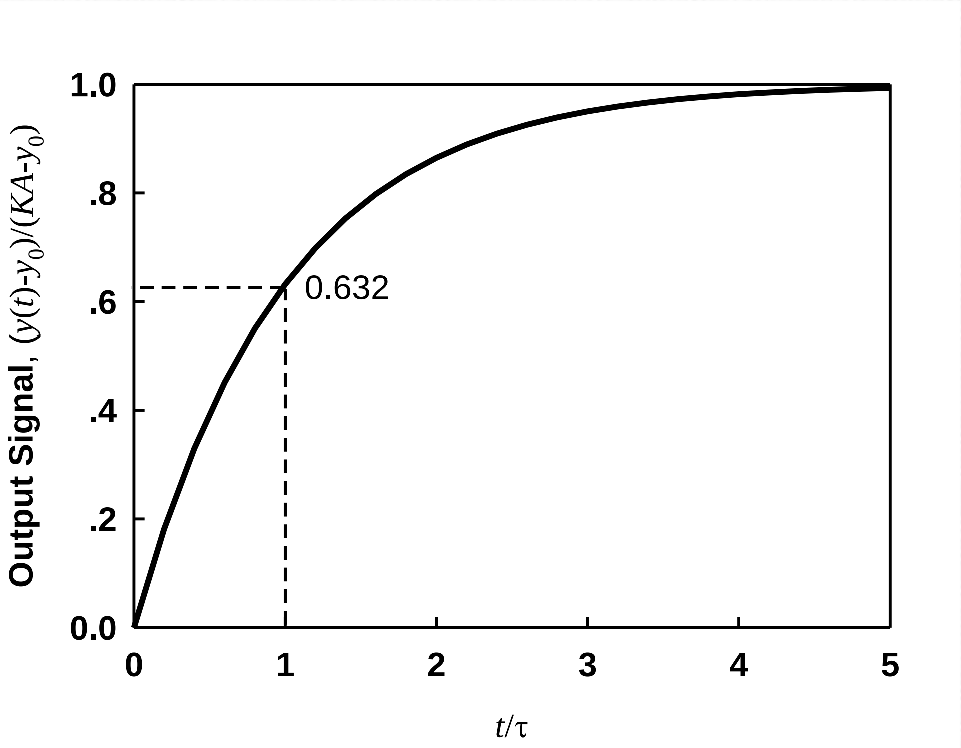
Applying the initial condition, we get $C = y_0$-KA, thus gives

$$\boxed{y(t) = KA + (y_0 - KA)e^{-t/\tau}}$$

First-Order Systems: Step Response

Non-dimensional step response of first-order instrument

$$\boxed{\frac{y(t)-y_0}{KA-y_0}=1-e^{-t/\tau}}$$



Time Response

- **Unit-Step Response of First-Order Systems.**

Since the Laplace transform of the **unit-step** function is **1/S**, substituting R(s) = 1/s into Equation

$$\frac{C(s)}{R(s)} = \frac{1}{Ts + 1} \qquad C(s) = \frac{1}{Ts + 1}\frac{1}{s}$$

Expanding C(s)into partial fractions gives

$$C(s) = \frac{1}{s} - \frac{T}{Ts + 1} = \frac{1}{s} - \frac{1}{s + (1/T)}$$

Taking the inverse Laplace transform of Equation (5-2), we obtain

$$c(t) = 1 - e^{-t/T}, \qquad \text{for } t \geq 0$$

Time Response

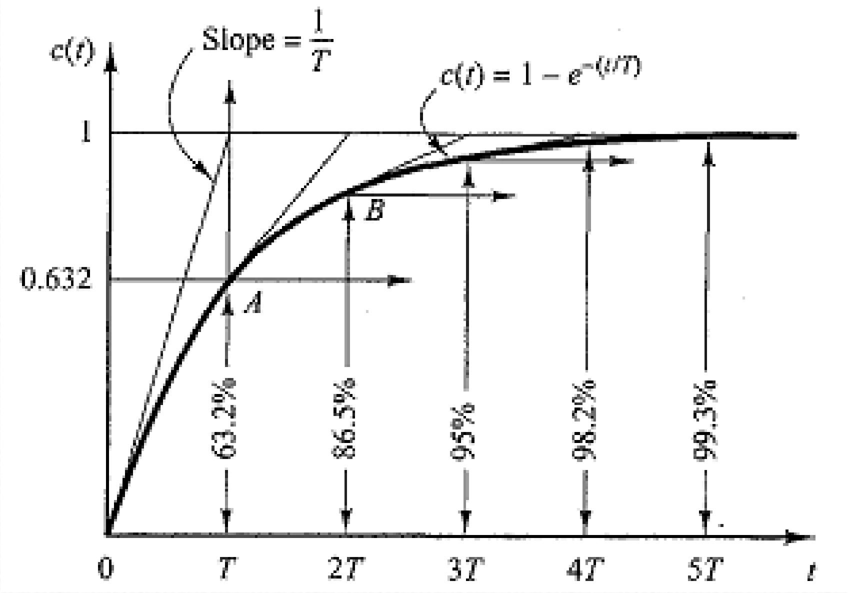
- **Unit-Step Response of First-Order Systems.**

One important characteristic of such an exponential response curve c(t)is that at t = T the value of c(t)is 0.632,or the response c(t)has reached 63.2% of its total change

$$c(T) = 1 - e^{-1} = 0.632$$

Time constant : T

Note that the **smaller** the time constant *T* ,the **faster** the system response.



Time Response

- **Unit-Ramp Response of First-Order Systems.**

Since the Laplace transform of the **unit-ramp** function is **1/S²**, we obtain the output of the system

$$C(s) = \frac{1}{Ts+1}\frac{1}{s^2}$$

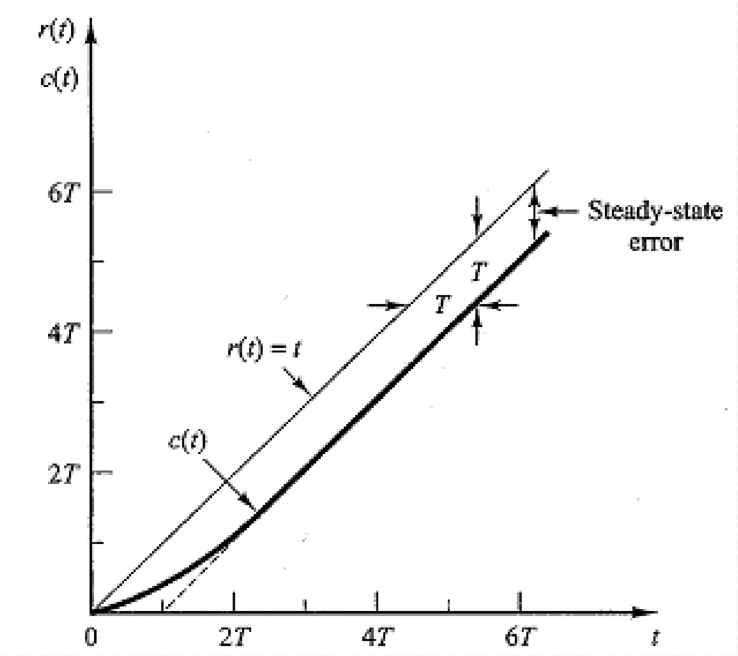
- Expanding C(s) into partial fractions gives

$$C(s) = \frac{1}{s^2} - \frac{T}{s} + \frac{T^2}{Ts+1}$$

- Taking the inverse Laplace transform of Equation, we obtain

$$c(t) = t - T + Te^{-t/T}, \qquad \text{for } t \geq 0$$

- The **smaller** the time constant T, the **smaller** the steady-state error in following the **ramp** input



Time Response

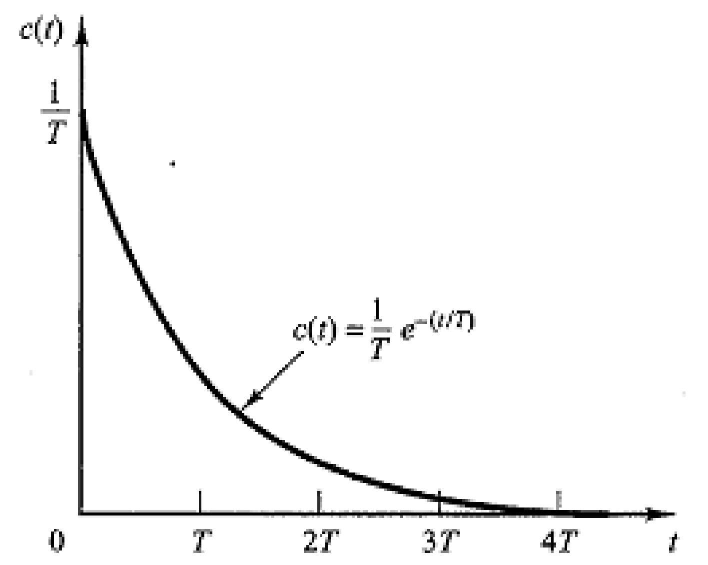
- **Unit-Impulse Response of First-Order Systems.**

For the **unit-impulse** input, ***R(s)*= 1** and the output of the system can be obtained as

$$C(s) = \frac{1}{Ts + 1}$$

- The inverse Laplace transform of Equation gives, and The response curve given by Equation is shown in Figure

$$c(t) = \frac{1}{T} e^{-t/T}, \qquad \text{for } t \geq 0$$



Time Response

Important Property of Linear Time-Invariant Systems.

- In the analysis above, it has been shown that for the **unit-ramp** input the output *c (t)* is

$$c(t) = t - T + Te^{-t/T}, \qquad \text{for } t \geq 0$$

- For the **unit-step** input, which is the **derivative** of **unit-ramp** input, the output *c (t)* is

$$c(t) = 1 - e^{-t/T}, \qquad \text{for } t \geq 0$$

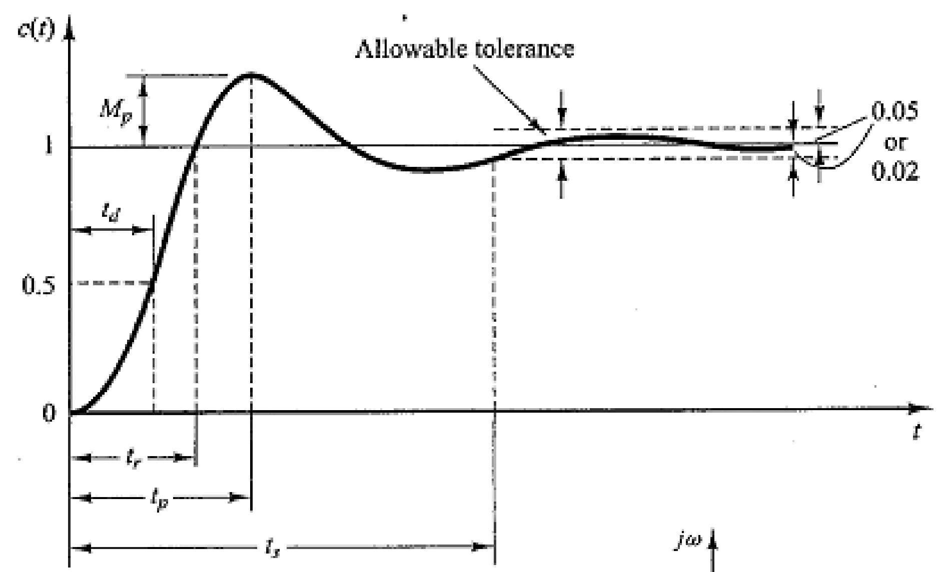
- Finally, for the **unit-impulse** input, which is the **derivative** of **unit-step** input, the output *c (t)* is

$$c(t) = \frac{1}{T}e^{-t/T}, \qquad \text{for } t \geq 0$$

Time Response

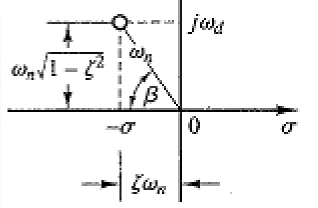
Second-order systems

Frequently, the performance characteristics of a control system are specified in terms of the transient response to a unit-step input



1. **Delay** time, **td**: The delay time is the time required for the response to reach half the final value the very first time. Since the peak time corresponds to the first peak overshoot,
2. **Rise** time, **t***r:* The rise time is the time required for the response to rise from 10% to 90%, 5% to 95%,or 0% to 100% of its final value. the 10%to 90%rise time is commonly used. Clearly, for a small value of *tr,* ω_n must be large.

$$t_r = \frac{\pi - \beta}{\omega_d}$$

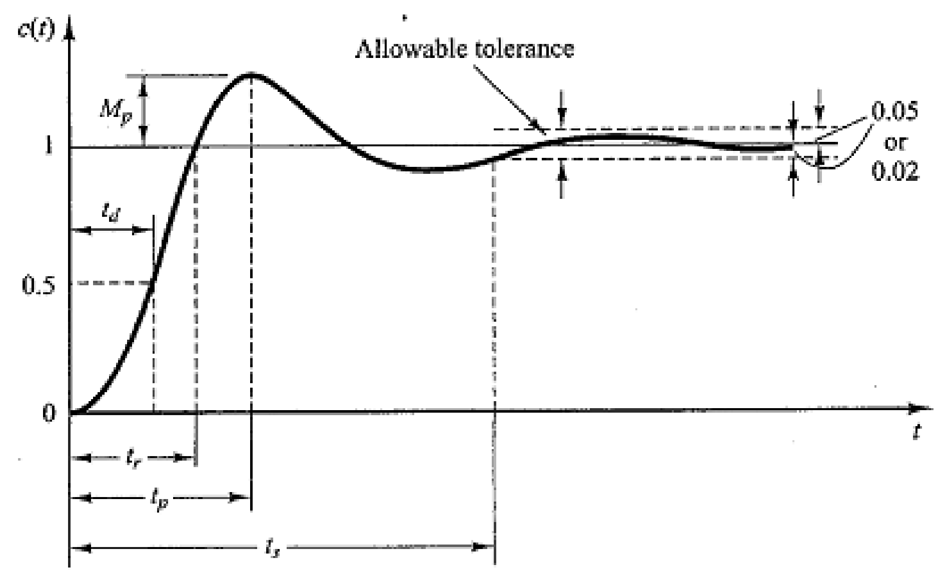


Time Response

Second-order systems

Frequently, the performance characteristics of a control system are specified in terms of the transient response to a unit-step input

3. **Peak** time, **tp**: The peak time is the time required for the response to reach the first peak of the overshoot. $t_p = \frac{\pi}{\omega_d}$
4. **Maximum** (percent) **overshoot**, **Mp**: The maximum overshoot is the maximum peak value of the response curve measured from unity. $M_p \; e^{-(\zeta/\sqrt{1-\zeta^2})\pi}$
5. **Settling** time, ***ts:*** The settling time is the time required for the response curve to reach and stay within a range about the final value. The settling time corresponding to ±2% or ± *5%* tolerance band may be measured in terms of the time constant $T = 1/\zeta\omega_n$



$$t_s = 4T = \frac{4}{\sigma} = \frac{4}{\zeta\omega_n} \qquad \text{(2\% criterion)}$$

$$t_s = 3T = \frac{3}{\sigma} = \frac{3}{\zeta\omega_n} \qquad \text{(5\% criterion)}$$

Time Response

Second-order systems

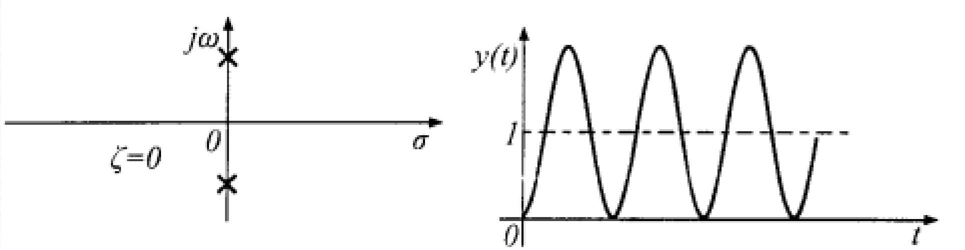
Case 1 $(\zeta = 0)$

In this case the poles of G(s) are imaginary since $s_{1,2} = \pm j\omega_n$, and relation becomes $Y(s) = \frac{\omega_n^2}{s(s^2+\omega_n^2)}$

If we expand Y(s) in partial fractions, we have

$$Y(s) = \frac{1}{s} - \frac{s}{s^2+\omega_n^2}, \qquad \text{and thus } y(t) = 1 - \cos\omega_n t$$

observe that the response y(t) is a sustained oscillation with constant frequency to ω_n and constant amplitude equal to 1. In this case, we say that the system is undamped



$\sigma = \omega_n\zeta$ and $\omega_d = \omega_n\sqrt{1-\zeta^2}$
are called the attenuation or damping constant and the damped natural frequency of the system, respectively

Time Response

Second-order systems

Case 2 $(0 < \zeta < 1)$

In this case the poles of G(s) are a complex conjugate pair since $s_{1,2} = -\sigma \pm j\omega_d$

, and relation becomes

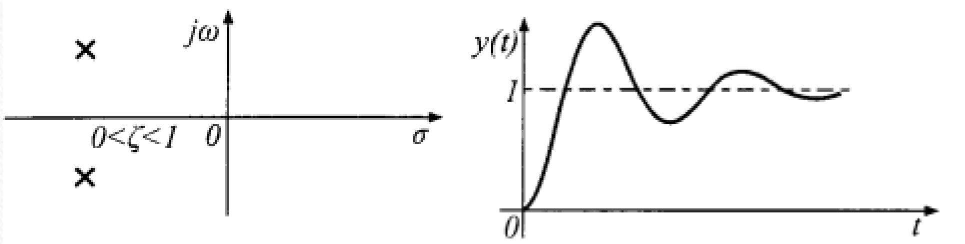
$$Y(s) = \frac{\omega_n^2}{s[(s+\sigma)^2 + \omega_d^2]}$$

If we expand Y(s) in partial fractions, we have

$$Y(s) = \frac{1}{s} - \frac{s+2\sigma}{(s+\sigma)^2 + \omega_d^2} = \frac{1}{s} - \frac{s+\sigma}{(s+\sigma)^2 + \omega_d^2} - \left[\frac{\sigma}{\omega_d}\right]\left[\frac{\omega_d}{(s+\sigma)^2 + \omega_d^2}\right]$$

observe that the response y(t) is a damped oscillation which tends to 1 as t → ∞.

In this case, we say that the system is underdamped.



$$\sigma = \omega_n \zeta \text{ and } \omega_d = \omega_n \sqrt{1-\zeta^2}$$

are called the attenuation or damping constant and the damped natural frequency of the system, respectively

$$\varphi = \tan^{-1} \frac{\sqrt{1-\zeta^2}}{\zeta}$$

$$y(t) = 1 - e^{-\sigma t} \cos \omega_d t - \frac{\sigma}{\omega_d} e^{-\sigma t} \sin \omega_d t \quad = 1 - \frac{e^{-\sigma t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \varphi), \quad = 1 - \frac{e^{-\sigma t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \varphi).$$

Time Response

Second-order systems

Case 3 $(\zeta = 1)$

In this case the poles of G(s) are the real double pole $-\omega_n$, and relation becomes

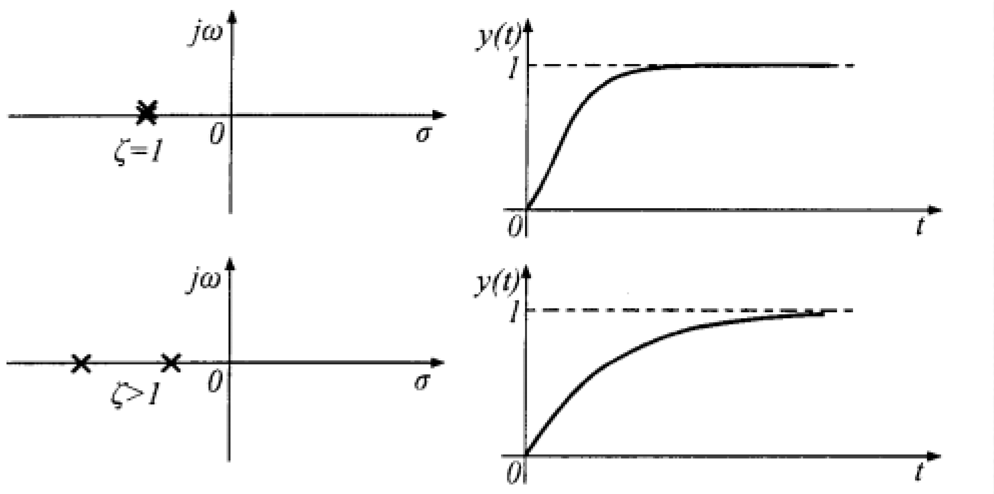
$$Y(s) = \frac{\omega_n^2}{s(s+\omega_n)^2}$$

If we expand Y(s) in partial fractions, we have

$$Y(s) = \frac{1}{s} - \frac{1}{s+\omega_n} - \frac{\omega_n^2}{(s+\omega_n)^2}, \qquad \text{and thus } y(t) = 1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t}$$

we observe that the waveform of the response y(t) involves no oscillations, and asymptotically tends to 1 as $t \to \infty$.

In this case we say that the system is critically damped



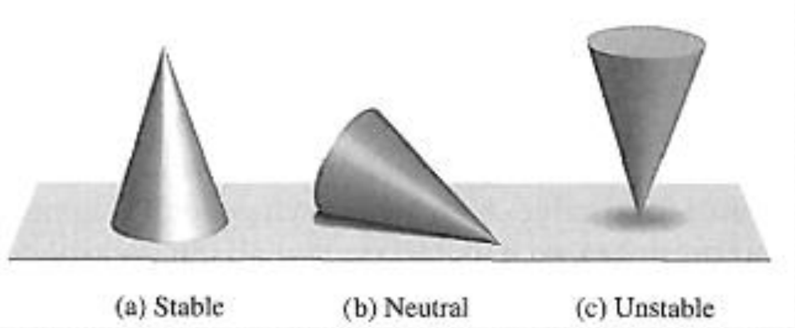
Model Examples

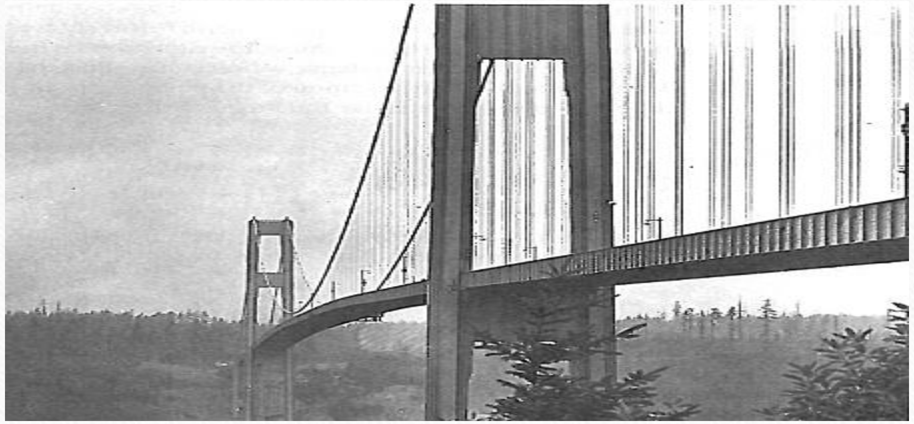
- DC- Motor Controller
- Response of 2^{nd} order system without controller

Stability Systems of Linear system

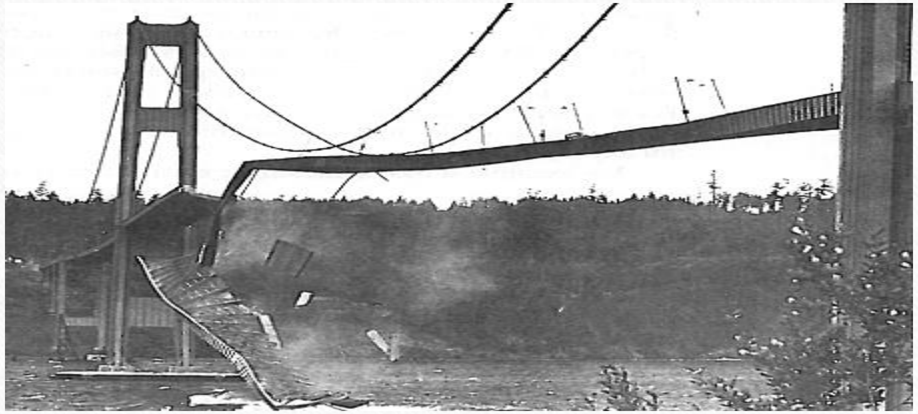
Definition :

The stability of a dynamic system is defined in a similar manner. The response to a displacement, or initial condition, will result in either a decreasing, neutral, or increasing response.





Stability of system after exciting force



Stability Systems of Linear system

Definition :

The characteristics equation of second order system

$$as^2 + bs + c = 0$$

The roots of the characteristic equation given in equation

$$s_1,\ s_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

These roots determine the transient response of the system and for a second-order system can be written as

a) Overdamping $s_1 = -\sigma_1$ $s_2 = -\sigma_2$

b) Critical damping $s_1 = s_2 = -\sigma$

c) underdamping $s_1,\ s_2 = -\sigma \pm \mathrm{j}\omega$ $s_1,\ s_2 = +\sigma \pm \mathrm{j}\omega$

(a) Stable (b) Unstable

Generally, if any of the roots of the characteristics equation have positive real parts, then the system will be unstable

It was stated that a control system is stable if and only if all closed-loop poles lie in the left-half s plane

Stability Systems of Linear system

Routh's Stability Criterion

absolute stability can be obtained directly from the coefficients of the characteristic equation.

The procedure in Routh's stability criterion is as follows:

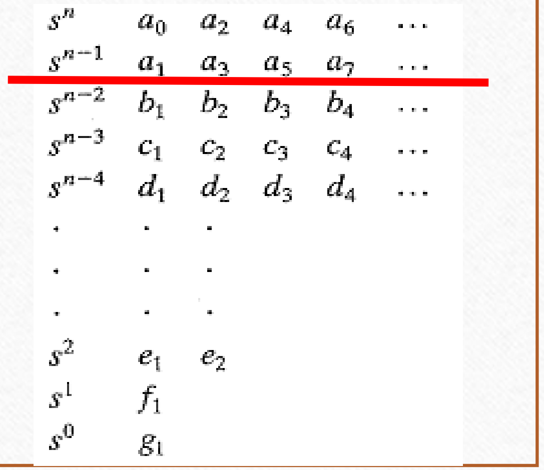
1. Write the polynomial in s in the following form:

$$a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n = 0$$

where the coefficients are real quantities. Assume that ***a* ≠ 0; that is, any zero root has been removed.**

2. If any of the coefficients are zero or negative in the presence of at least one positive coefficient, there is a root or roots that are imaginary or that have positive

3. If all coefficients are positive, arrange the coefficients of the polynomial in rows and columns according to the following pattern:



s^n	a_0	a_2	a_4	a_6	$\cdots$
s^{n-1}	a_1	a_3	a_5	a_7	$\cdots$
s^{n-2}	b_1	b_2	b_3	b_4	$\cdots$
s^{n-3}	c_1	c_2	c_3	c_4	$\cdots$
s^{n-4}	d_1	d_2	d_3	d_4	$\cdots$
.	.	.			
.	.	.			
.	.	.			
s^2	e_1	e_2			
s^1	f_1				
s^0	g_1				

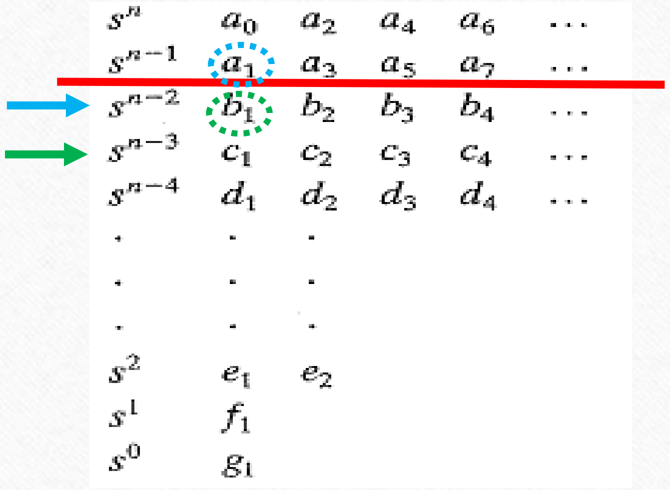
Stability Systems of Linear system

Routh's Stability Criterion

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1} \qquad c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1}$$

$$b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1} \qquad c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1}$$

$$b_3 = \frac{a_1 a_6 - a_0 a_7}{a_1} \qquad c_3 = \frac{b_1 a_7 - a_1 b_4}{b_1}$$



s^n	a_0	a_2	a_4	a_6	$\cdots$
s^{n-1}	a_1	a_3	a_5	a_7	$\cdots$
s^{n-2}	b_1	b_2	b_3	b_4	$\cdots$
s^{n-3}	c_1	c_2	c_3	c_4	$\cdots$
s^{n-4}	d_1	d_2	d_3	d_4	$\cdots$
.	.	.			
.	.	.			
.	.	.			
s^2	e_1	e_2			
s^1	f_1				
s^0	g_1				

The Routh-Hurwitz criterion states that the number of roots of *q(s)* with positive real parts is equal to the number of changes in sign of the first column of the Routh array

The number of changes of sign in the first column of the array developed for the polynomial in s equal to the number of roots that are located to the right of the vertical line

Model Examples

- Pulse Width Modulation (PWM)